

# The compositional inverse of a class of linearized permutation polynomials over $\mathbb{F}_{2^n}$ , $n$ odd\*

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## Abstract

In this paper, the compositional inverses of a class of linearized permutation polynomials of the form  $P(x) = x + x^2 + \text{tr}\left(\frac{x}{a}\right)$  over the finite field  $\mathbb{F}_{2^n}$  for an odd positive integer  $n$  are explicitly determined.

**Keywords** Permutation polynomial; Linearized polynomial; Compositional inverse; Determinant.

## 1 Introduction

Let  $\mathbb{F}_q$  be the finite field with  $q$  elements where  $q$  is a prime or a prime power. A polynomial over  $\mathbb{F}_q$  is called a permutation polynomial (or a PP for short) if the map

$$\begin{aligned} \mathbb{F}_q &\longrightarrow \mathbb{F}_q \\ c &\longmapsto f(c) \end{aligned}$$

is bijective [4]. It is clear that the set of all PPs over  $\mathbb{F}_q$  forms a group under the operation of composition of polynomials and subsequent reduction modulo  $(x^q - x)$ , which is isomorphic to  $\mathcal{S}_q$ , the symmetric group on  $q$  letters.

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Hence for every PP over  $\mathbb{F}_q$ , there exists a unique polynomial  $f^{-1}(x)$  over  $\mathbb{F}_q$  such that  $f(f^{-1}(x)) = f^{-1}(f(x)) = x$  in the sense of reducing modulo  $(x^q - x)$ , called the compositional inverse of  $f(x)$ . Generally speaking, for a given PP  $f(x)$  over  $\mathbb{F}_q$ , it is easy to verify whether the other polynomial  $g(x)$  is its compositional inverse since we need only to check whether  $f(g(x)) \equiv x \pmod{(x^q - x)}$  or  $g(f(x)) \equiv x \pmod{(x^q - x)}$  holds. But to derive the compositional inverse of a PP explicitly is very challenging. Indeed, it was posed by G.L. Mullen as a research problem that to compute the coefficients of the inverse polynomial of a PP efficiently [6, Problem 10]. Up to present, there are only a few class of PPs over finite fields whose compositional inverses can be explicitly obtained (see e.g. [5, 3, 7]). For instance, a method to compute compositional inverses of linearized permutation polynomials was proposed by the authors in [7].

Linearized polynomials are a special class of polynomials over finite fields. A linearized polynomial over the finite field  $\mathbb{F}_{q^n}$  is of the form

$$L(x) = \sum_{i=0}^{n-1} a_i x^{q^i}.$$

It can induce a linear transformation of  $\mathbb{F}_{q^n}$  over  $\mathbb{F}_q$  due to the well-known relation  $(x+y)^{q^t} = x^{q^t} + y^{q^t}$  for any  $x, y \in \mathbb{F}_{q^n}$  and  $0 \leq t \leq n-1$ . A criterion on linearized polynomials to be permutation ones is easy to obtain. In fact, it was found by Dickson as early as in 1897 that  $L(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$  is a PP if and only if the matrix

$$D_L = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1}^q & a_0^q & \dots & a_{n-2}^q \\ \vdots & \vdots & & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \dots & a_0^{q^{n-1}} \end{pmatrix}$$

is non-singular [4]. Obviously, the compositional inverse of a linearized PP is also a linearized polynomial.

In [7],  $D_L$  is called the associate Dickson matrix of the linearized polynomial  $L(x)$ , or vice versa, by the authors. We have also found more relations between the properties of a linearized polynomial and its associate Dickson matrix. For example, we have related the compositional inverse of a linearized PP to the inverse of its associate Dickson matrix, obtaining the following theorem:

**Theorem 1.1.** *Let  $L(x) \in \mathbb{F}_{q^n}[x]$  be a linearized PP. Then  $D_{L^{-1}} = D_L^{-1}$ . More precisely, if  $L(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$  and  $\bar{a}_i$  is the  $(i, 0)$ -th cofactor of  $a_i$ ,  $0 \leq i \leq n-1$ , then*

$$L^{-1}(x) = \frac{1}{\det L} \sum_{i=0}^{n-1} \bar{a}_i x^{q^i},$$

where  $\det L = \sum_{i=0}^{n-1} a_{n-i}^{q^i} \bar{a}_i \in \mathbb{F}_q$  (subscripts reduced modulo  $n$ ).

According to Theorem 1.1, the main task in computing the compositional inverse of a linearized PP over  $\mathbb{F}_{q^n}$  is to compute the determinant of  $n$  matrices over  $\mathbb{F}_{q^n}$  with size  $(n-1) \times (n-1)$ , which can be efficiently realized from an algorithmic perspective. However, to explicitly represent the compositional inverses of certain linearized PPs by closed form formulas seems not so direct due to the difficulties in computing determinants, even for some linearized PPs of simple forms. In this paper, we devote to talk about such an example.

Let  $n$  be odd and  $a \in \mathbb{F}_{2^n}$  be a non-zero element with  $\text{tr}\left(\frac{1}{a}\right) = 1$ . It can be easily derived by [2, Theorem 7] that

$$P(x) = x + x^2 + \text{tr}\left(\frac{x}{a}\right)$$

is a linearized PP over  $\mathbb{F}_{2^n}$ , where “tr” is the trace map over  $\mathbb{F}_{2^n}$ , i.e.  $\text{tr}(x) = \sum_{i=0}^{n-1} x^{2^i}$  for any  $x \in \mathbb{F}_{2^n}$ . Linearized PPs of this form can be related to the multiplication of the binary Knuth pre-semifield [1].

In the subsequent sections, we propose the explicit representation of coefficients of the compositional inverse of  $P(x)$  and explain how to obtain it afterwards. It will be noted that  $P^{-1}(x)$  is of a very complicate form.

## 2 Explicit representation of the compositional inverse of $P(x)$

To simplify the presentation of coefficients of  $P^{-1}(x)$ , we introduce a symbol firstly.

For any  $c \in \mathbb{F}_{2^n}$  and  $i_1, \dots, i_t \in \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ , we denote

$$c\langle i_1, i_2, \dots, i_t \rangle = \sum_{j=1}^t c^{2^{i_j}}.$$

Besides, for any subset  $I$  of  $\mathbb{Z}/n\mathbb{Z}$ , we denote

$$c\langle I \rangle = \sum_{i \in I} c^{2^i}.$$

It is easy to see that the symbol “ $c\langle \cdot \rangle$ ” satisfies the following properties:

- Proposition 2.1.** (1)  $c\langle 0, 1, \dots, n-1 \rangle = c\langle \mathbb{Z} \rangle = \text{tr}(c)$ ;  
(2) For any  $\pi \in \mathcal{S}_t$ ,  $c\langle i_{\pi(1)}, i_{\pi(2)}, \dots, i_{\pi(t)} \rangle = c\langle i_1, i_2, \dots, i_t \rangle$ ;  
(3) For any  $I, J \subseteq \mathbb{Z}/n\mathbb{Z}$ ,  $c\langle I \rangle + c\langle J \rangle = c\langle (I \cup J) \setminus (I \cap J) \rangle$ ;  
(4) For any  $k \geq 0$ ,  $c\langle i_1, i_2, \dots, i_t \rangle^{2^k} = c\langle i_1 + k, i_2 + k, \dots, i_t + k \rangle$ .  $\square$

By this newly defined symbol, we can propose the compositional inverse of  $P(x)$ .

**Theorem 2.2.**  $P^{-1}(x) = B_a(x) + \text{tr}(x)$ , where the coefficients of  $B_a(x) = \sum_{i=0}^{n-1} b_i x^{2^i} \in \mathbb{F}_{2^n}[x]$  are given by

$$\begin{aligned} b_0 &= \frac{1}{a} \langle 2, 4, 6, \dots, n-1 \rangle, \\ b_i &= \begin{cases} \frac{1}{a} \langle 1, 3, 5, \dots, i, i+1, i+3, \dots, n-1 \rangle & \text{if } i \text{ is odd} \\ \frac{1}{a} \langle 1, 3, 5, \dots, i-1, i+2, i+4, \dots, n-1 \rangle & \text{if } i \text{ is even,} \end{cases} \end{aligned}$$

$$1 \leq i \leq n-1$$

*Proof.* Noting that

$$\begin{aligned} B_a(1) &= \sum_{i=0}^{n-1} b_i \\ &= \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle + \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{1}{a} \langle 1, 3, \dots, i, i+1, i+3, \dots, n-1 \rangle \\ &\quad + \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ even}}} \frac{1}{a} \langle 1, 3, \dots, i-1, i+2, i+4, \dots, n-1 \rangle \\ &= \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle + \sum_{j=1}^{\frac{n-1}{2}} \left( \frac{1}{a} \langle 1, 3, \dots, 2j-1, 2j, 2j+2, \dots, n-1 \rangle \right. \\ &\quad \left. + \frac{1}{a} \langle 1, 3, \dots, 2j-1, 2j+2, 2j+4, \dots, n-1 \rangle \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle + \sum_{j=1}^{\frac{n-1}{2}} \frac{1}{a} \langle 2j \rangle \\
&= 0,
\end{aligned}$$

we get

$$\begin{aligned}
L^{-1}(L(x)) &= B_a(x) + B_a(x^2) + B_a(1) \operatorname{tr} \left( \frac{x}{a} \right) + \operatorname{tr}(x + x^2) + \operatorname{tr} \left( \frac{x}{a} \right) \\
&= B_a(x) + B_a(x^2) + \operatorname{tr} \left( \frac{x}{a} \right).
\end{aligned}$$

Furthermore, by Proposition 2.1 we have

$$\begin{aligned}
B_a(x) + B_a(x^2) &= \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle x + \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{1}{a} \langle 1, 3, \dots, i, i+1, i+3, \dots, n-1 \rangle x^{2^i} \\
&\quad + \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ even}}} \frac{1}{a} \langle 1, 3, \dots, i-1, i+2, i+4, \dots, n-1 \rangle x^{2^i} \\
&\quad + \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle x^2 + \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{1}{a} \langle 1, 3, \dots, i, i+1, i+3, \dots, n-1 \rangle x^{2^{i+1}} \\
&\quad + \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ even}}} \frac{1}{a} \langle 1, 3, \dots, i-1, i+2, i+4, \dots, n-1 \rangle x^{2^{i+1}} \\
&= \left( \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle + \frac{1}{a} \langle 1, 3, \dots, n-2 \rangle \right) x \\
&\quad + \left( \frac{1}{a} \langle 1, 2, 4, \dots, n-1 \rangle + \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle \right) x^2 \\
&\quad + \sum_{\substack{3 \leq i \leq n-1 \\ i \text{ odd}}} \left( \frac{1}{a} \langle 1, 3, \dots, i, i+1, i+3, \dots, n-1 \rangle \right. \\
&\quad \quad \left. + \frac{1}{a} \langle 1, 3, \dots, i-2, i+1, i+3, \dots, n-1 \rangle \right) x^{2^i} \\
&\quad + \sum_{\substack{2 \leq i \leq n-1 \\ i \text{ even}}} \left( \frac{1}{a} \langle 1, 3, \dots, i-1, i+2, i+4, \dots, n-1 \rangle \right. \\
&\quad \quad \left. + \frac{1}{a} \langle 1, 3, \dots, i-1, i, i+2, \dots, n-1 \rangle \right) x^{2^i}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a}\langle 1, 2, 3, 4, \dots, n-1 \rangle x + \frac{1}{a}\langle 1 \rangle x^2 + \sum_{2 \leq i \leq n-1} \frac{1}{a}\langle i \rangle x^{2^i} \\
&= \left(x + \frac{x}{a}\right) + \sum_{1 \leq i \leq n-1} \left(\frac{x}{a}\right)^{2^i} \\
&= x + \text{tr}\left(\frac{x}{a}\right).
\end{aligned}$$

Then  $L^{-1}(L(x)) = x$  follows.  $\square$

**Remark 2.3.** Assume  $P^{-1}(x) = \sum_{i=0}^{n-1} p_i x^{2^i}$ . Then it is easy to derive from Theorem 2.2 and the condition  $\text{tr}\left(\frac{1}{a}\right) = 1$  that

$$\begin{aligned}
p_0 &= 1 + \frac{1}{a}\langle 2, 4, \dots, n-1 \rangle \\
&= \frac{1}{a}\langle 0, 1, 3, \dots, n-2 \rangle,
\end{aligned}$$

and for  $1 \leq i \leq n-1$ ,

$$\begin{aligned}
p_i &= 1 + \frac{1}{a}\langle 1, 3, \dots, i, i+1, i+3, \dots, n-1 \rangle \\
&= \frac{1}{a}\langle 0, 2, 4, \dots, i-1, i+2, i+4, \dots, n-2 \rangle
\end{aligned}$$

when  $i$  is odd, and

$$\begin{aligned}
p_i &= 1 + \frac{1}{a}\langle 1, 3, \dots, i-1, i+2, i+4, \dots, n-1 \rangle \\
&= \frac{1}{a}\langle 0, 2, 4, \dots, i, i+1, i+3, \dots, n-2 \rangle.
\end{aligned}$$

when  $i$  is even. Therefor, we can also represent  $P^{-1}(x)$  by  $P^{-1}(x) = C_a(x) + \text{tr}\left(\frac{x}{a}\right)$ , where the coefficients of  $C_a(x) = \sum_{i=0}^{n-1} c_i x^{2^i} \in \mathbb{F}_{2^n}[x]$  are given by

$$\begin{aligned}
c_0 &= \frac{1}{a}\langle 1, 3, 5, \dots, n-3 \rangle, \\
c_i &= \begin{cases} 1 + \frac{1}{a}\langle 1, 3, 5, \dots, i-2, i+1, i+3, \dots, n-1 \rangle & \text{if } i \text{ is odd} \\ \frac{1}{a}\langle 0, 2, 4, \dots, i-2, i+1, i+3, \dots, n-2 \rangle & \text{if } i \text{ is even,} \end{cases}
\end{aligned}$$

$1 \leq i \leq n-1$ .

**Corollary 2.4.** *Let  $P_1(x) = x + x^2 + \text{tr}(x)$ . Then  $P_1(x)$  is a linearized PP over  $\mathbb{F}_{2^n}$  with compositional inverse*

$$P_1^{-1}(x) = \begin{cases} \sum_{i=0}^{\frac{n-1}{2}} x^{2^{2i}} & \text{if } n \equiv 1 \pmod{4} \\ \sum_{i=0}^{\frac{n-3}{2}} x^{2^{2i+1}} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

*Proof.* It is obvious that  $P_1(x)$  is a linearized PP over  $\mathbb{F}_{2^n}$  since  $\text{tr}(1) = 1$ . By fixing  $a = 1$  in Theorem 2.2, it is straightforward to obtain the coefficients of  $P_1^{-1}(x)$ .  $\square$

**Remark 2.5.** (1) *It is easy to see that the compositional inverse of  $P_1(x)$  in Corollary 2.4 can be written as*

$$P_1^{-1}(x) = \frac{n+1}{2} \sum_{i=0}^{\frac{n-1}{2}} x^{2^{2i}} + \frac{n-1}{2} \sum_{i=0}^{\frac{n-3}{2}} x^{2^{2i+1}}.$$

(2) *In fact,  $P_1(x)$  is a linearized polynomial with coefficients in  $\mathbb{F}_2$ , whose conventional associate [4] is  $p_1(x) = \sum_{i=2}^{n-1} x^i$ . By [4, Lemma 3.59], it is easy to derive that the conventional associate of  $P_1^{-1}(x)$  is just the polynomial  $\bar{p}(x)$  satisfying  $p(x)\bar{p}(x) \equiv 1 \pmod{(x^n + 1)}$ . Hence from Corollary 2.4, we obtain*

$$\bar{p}(x) = \begin{cases} \sum_{i=0}^{\frac{n-1}{2}} x^{2^i} & \text{if } n \equiv 1 \pmod{4} \\ \sum_{i=0}^{\frac{n-3}{2}} x^{2^{i+1}} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

### 3 The method to obtain Theorem 2.2

In this section, we describe in detail the process to derive the compositional inverse of  $P(x)$ . Note that the associate Dickson matrix of  $P(x)$  is

$$D_P = \begin{pmatrix} 1 + \frac{1}{a} & 1 + \frac{1}{a^2} & \frac{1}{a^{2^2}} & \cdots & \frac{1}{a^{2^{n-2}}} & \frac{1}{a^{2^{n-1}}} \\ \frac{1}{a} & 1 + \frac{1}{a^2} & 1 + \frac{1}{a^{2^2}} & \cdots & \frac{1}{a^{2^{n-2}}} & \frac{1}{a^{2^{n-1}}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{a} & \frac{1}{a^2} & \frac{1}{a^{2^2}} & \cdots & 1 + \frac{1}{a^{2^{n-2}}} & 1 + \frac{1}{a^{2^{n-1}}} \\ 1 + \frac{1}{a} & \frac{1}{a^2} & \frac{1}{a^{2^2}} & \cdots & \frac{1}{a^{2^{n-2}}} & 1 + \frac{1}{a^{2^{n-1}}} \end{pmatrix},$$

which is of a relatively complicate form. To utilizing Theorem 1.1 easily, we firstly simplify the problem by a linear transformation of  $P(x)$ . That is, we reduce the problem of computing compositional inverse of  $P(x)$  to that of computing the compositional inverse of  $\tilde{P}(x) = ax + a^2x^2 + \text{tr}(x)$  by noting that  $\tilde{P}(x) = P(ax)$ , which implies

$$P^{-1}(x) = a\tilde{P}^{-1}(x). \quad (1)$$

In this place, we should deal with the Dickson matrix

$$D_{\tilde{P}} = \begin{pmatrix} 1 + a & 1 + a^2 & 1 & \cdots & 1 & 1 \\ 1 & 1 + a^2 & 1 + a^{2^2} & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 + a^{2^{n-2}} & 1 + a^{2^{n-1}} \\ 1 + a & 1 & 1 & \cdots & 1 & 1 + a^{2^{n-1}} \end{pmatrix},$$

which seems much simpler than  $D_P$ .

Assume the  $(i, 0)$ -th cofactor of  $D_{\tilde{P}}$  is  $\tilde{p}_i$ ,  $0 \leq i \leq n-1$ . Then

$$\det \tilde{P} = (1 + a)\tilde{p}_0 + \sum_{i=1}^{n-2} \tilde{p}_i + (1 + a)\tilde{p}_{n-1}.$$

However, we can deduce  $\det \tilde{P} = 1$  before computing  $\tilde{p}_i$ ,  $0 \leq i \leq n-1$ , since  $\det \tilde{P}$  is a non-zero element of  $\mathbb{F}_2$  according to Theorem 2.2. To explicitly represent  $\tilde{p}_i$ ,  $0 \leq i \leq n-1$ , we give the following lemma on determinants of matrices of a special form firstly.



**Lemma 3.1.** *Let  $a_1, \dots, a_m, b_1, \dots, b_{m-1} \in R$  where  $R$  is a commutative ring with identity and  $m \geq 2$  is a positive integer. Then*

$$\det \begin{pmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{m-1} & b_{m-1} \\ 1 & 1 & \dots & 1 & a_m \end{pmatrix} = (-1)^{m+1} \prod_{j=1}^{m-1} b_j + \sum_{i=1}^{m-2} (-1)^{m+i+1} \prod_{j=1}^i \prod_{k=i+1}^{m-1} a_j b_k + \prod_{j=1}^m a_j.$$

*Proof.* The determinant can be directly computed using Laplace expansion along the last row of the matrix.  $\square$

To simplify the computations of  $\tilde{p}_i$ ,  $0 \leq i \leq n-1$ , we introduce another symbol and propose several obvious properties of it.

For any  $c \in \mathbb{F}_{2^n}^*$ ,  $i_1, \dots, i_t \in \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$  and  $I \subseteq \mathbb{Z}/n\mathbb{Z}$ , we denote

$$c[i_1, i_2, \dots, i_t] = \prod_{j=1}^t c^{2^{i_j}} = c^{2^{i_1} + \dots + 2^{i_t}}$$

and

$$c[I] = \prod_{i \in I} c^{2^i}.$$

**Proposition 3.2.** (1)  $c[0, 1, \dots, n-1] = N(c) = 1$ , where “ $N$ ” is the norm map over  $\mathbb{F}_{2^n}$ ;

(2) For any  $\pi \in \mathcal{S}_t$ ,  $c[i_{\pi(1)}, i_{\pi(2)}, \dots, i_{\pi(t)}] = c[i_1, i_2, \dots, i_t]$ ;

(3) For any  $I, J \subseteq \mathbb{Z}/n\mathbb{Z}$ ,  $c[I]c[J] = c[I \cup J]c[I \cap J] = c[(I \cup J) \setminus (I \cap J)]c[I \cap J]^2$ ;

(4) For any  $k \geq 0$ ,  $c[i_1, i_2, \dots, i_t]^{2^k} = c[i_1 + k, i_2 + k, \dots, i_t + k]$ ;

(5) For any  $i \subseteq \mathbb{Z}/n\mathbb{Z}$ ,  $c[i] = c\langle i \rangle$ .  $\square$

Based on these preparations,  $\tilde{p}_i$ ,  $0 \leq i \leq n-1$ , are computed in the following Lemmas.

**Lemma 3.3.**

$$\tilde{p}_0 = \frac{1}{a} \left( 1 + \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle \right).$$

*Proof.*

$$\begin{aligned}
\tilde{p}_0 &= \det \begin{pmatrix} 1+a[1] & 1+a[2] & 1 & \cdots & 1 & 1 \\ 1 & 1+a[2] & 1+a[3] & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1+a[n-2] & 1+a[n-1] \\ 1 & 1 & 1 & \cdots & 1 & 1+a[n-1] \end{pmatrix} \\
&= \det \begin{pmatrix} a[1] & & a[3] & & & \\ & a[2] & & a[4] & & \\ & & \ddots & & \ddots & \\ & & & a[n-3] & & a[n-1] \\ & & & & a[n-2] & \\ 1 & 1 & \cdots & 1 & 1 & 1+a[n-1] \end{pmatrix}.
\end{aligned}$$

By Laplace expansion we get

$$\tilde{p}_0 = a[1]a[3] \cdots a[n-4]a[n-2] \cdot \det \begin{pmatrix} a[2] & a[4] & & & \\ & a[4] & a[6] & & \\ & & \ddots & \ddots & \\ & & & a[n-3] & a[n-1] \\ 1 & 1 & \cdots & 1 & 1+a[n-1] \end{pmatrix}.$$

Then from Lemma 3.1 and Proposition 3.2 we obtain

$$\begin{aligned}
\tilde{p}_0 &= a[1, 3, \dots, n-2] \cdot \left[ a[4, 6, \dots, n-1] + \sum_{j=1}^{\frac{n-5}{2}} a[2, 4, \dots, 2j]a[2j+4, 2j+6, \dots, n-1] \right. \\
&\quad \left. + a[2, 4, \dots, n-3](1+a[n-1]) \right] \\
&= \frac{N(a)}{a[0, 2]} + \sum_{j=1}^{\frac{n-5}{2}} \frac{N(a)}{a[0, 2j+2]} + \frac{N(a)}{a[0, n-1]}(1+a[n-1]) \\
&= \frac{1}{a} \left( \frac{1}{a[2]} + \sum_{j=1}^{\frac{n-5}{2}} \frac{1}{a[2j+2]} + \frac{1}{a[n-1]} + 1 \right) \\
&= \frac{1}{a} \left( 1 + \frac{1}{a} \langle 2, 4, \dots, n-1 \rangle \right).
\end{aligned}$$

□

**Lemma 3.4.**

$$\tilde{p}_1 = \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 2, 4, \dots, n-1 \rangle \right).$$

*Proof.*

$$\begin{aligned} \tilde{p}_1 &= \det \begin{pmatrix} 1+a[1] & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1+a[2] & 1+a[3] & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1+a[n-2] & 1+a[n-1] \\ 1 & 1 & 1 & \cdots & 1 & 1+a[n-1] \end{pmatrix} \\ &= \det \begin{pmatrix} a[1] & a[2] & a[3] & & & \\ & a[2] & & a[4] & & \\ & & a[3] & & a[5] & \\ & & & \ddots & & \ddots \\ & & & & a[n-3] & & a[n-1] \\ & & & & & a[n-2] & \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1+a[n-1] \end{pmatrix}. \end{aligned}$$

By Laplace expansion we get

$$\tilde{p}_1 = a[3]a[5] \cdots a[n-2] \cdot \det \begin{pmatrix} a[1] & a[2] & & & \\ & a[2] & a[4] & & \\ & & \ddots & \ddots & \\ & & & a[n-3] & a[n-1] \\ 1 & 1 & \cdots & 1 & 1+a[n-1] \end{pmatrix}.$$

Then from Lemma 3.1 and Proposition 3.2 we obtain

$$\begin{aligned} \tilde{p}_1 &= a[3, 5, \dots, n-2] \cdot \left[ a[2, 4, \dots, n-1] + a[1, 4, 6, \dots, n-1] \right. \\ &\quad + \sum_{j=2}^{\frac{n-3}{2}} a[1, 2, 4, \dots, 2j-2] a[2j+2, 2j+4, \dots, n-1] \\ &\quad \left. + a[1, 2, 4, \dots, n-3] (1 + a[n-1]) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{N(a)}{a[0,1]} + \frac{N(a)}{a[0,2]} + \sum_{j=2}^{\frac{n-3}{2}} \frac{N(a)}{a[0,2j]} + \frac{N(a)}{a[0,n-1]}(1 + a[n-1]) \\
&= \frac{1}{a} \left( \frac{1}{a[1]} + \frac{1}{a[2]} + \sum_{j=2}^{\frac{n-3}{2}} \frac{1}{a[2j]} + \frac{1}{a[n-1]} + 1 \right) \\
&= \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 2, 4, \dots, n-1 \rangle \right).
\end{aligned}$$

□

**Lemma 3.5.** For  $2 \leq i \leq n-3$ ,

$$\tilde{p}_i = \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, i-1, i+2, i+4, \dots, n-1 \rangle \right)$$

when  $i$  is even, and

$$\tilde{p}_i = \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, i, i+1, i+3, \dots, n-1 \rangle \right)$$

when  $i$  is odd.

*Proof.*

$$\begin{aligned}
\tilde{p}_i &= \det \begin{pmatrix} \frac{1+a[1]}{1+a[1]} & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \frac{1}{1+a[1]} & 1+a[2] & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a[i-1] & 1+a[i] & 1+a[i+1] & 1+a[i+2] & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1+a[n-2] & 1+a[n-1] \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1+a[n-1] \end{pmatrix} \\
&= \det \begin{pmatrix} & a[2] & & & & & & & & \\ a[1] & & a[3] & & & & & & & \\ & \ddots & & \ddots & & & & & & \\ & & a[i-2] & & a[i] & & & & & \\ & & & a[i-1] & a[i] & a[i+1] & a[i+2] & & & \\ & & & & a[i+1] & & a[i+3] & & & \\ & & & & & \ddots & & \ddots & & \\ & & & & & & a[n-3] & & a[n-1] & \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1+a[n-1] \end{pmatrix}.
\end{aligned}$$

(1) When  $i$  is even, we can get by Laplace expansion that

$$\tilde{p}_i = a[2, 4, \dots, i, i+1, i+3, \dots, n-2] \cdot \det \begin{pmatrix} a[1] & a[3] & & & & & & & & \\ & a[3] & a[5] & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & a[i-3] & a[i-1] & & & & & \\ & & & & a[i-1] & a[i+2] & & & & \\ & & & & & a[i+2] & a[i+4] & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & a[n-3] & a[n-1] & \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1+a[n-1] & \end{pmatrix}.$$

Then from Lemma 3.1 and Proposition 3.2 we obtain

$$\begin{aligned} \tilde{p}_i &= a[2, 4, \dots, i, i+1, i+3, \dots, n-2] \cdot \left[ a[3, 5, \dots, i-1, i+2, i+4, \dots, n-1] \right. \\ &\quad + \sum_{j=2}^{\frac{n-4}{2}} a[1, 3, \dots, 2j-1, 2j+3, \dots, i-1, i+2, i+4, \dots, n-1] \\ &\quad + a[1, 3, \dots, i-3, i+2, i+4, \dots, n-1] + a[1, 3, \dots, i-1, i+4, i+6, \dots, n-1] \\ &\quad + \sum_{j=2}^{\frac{n-i-3}{2}} a[1, 3, \dots, i-1, i+2, i+4, \dots, i+2j-2, i+2j+2, \dots, n-1] \\ &\quad \left. + a[1, 3, \dots, i-1, i+2, i+4, \dots, n-3](1 + a[n-1]) \right] \\ &= \frac{N(a)}{a[0, 1]} + \sum_{j=2}^{\frac{n-4}{2}} \frac{N(a)}{a[0, 2j+1]} + \frac{N(a)}{a[0, i-1]} + \frac{N(a)}{a[0, i+2]} \\ &\quad + \sum_{j=2}^{\frac{n-i-3}{2}} \frac{N(a)}{a[0, 2j]} + \frac{N(a)}{a[0, n-1]}(1 + a[n-1]) \\ &= \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, i-1, i+2, i+4, \dots, n-1 \rangle \right); \end{aligned}$$

(2) When  $i$  is odd, we can get by Laplace expansion that

$$\tilde{p}_i = a[2, 4, \dots, i-1, i+2, i+4, \dots, n-2] \cdot$$

$$\det \begin{pmatrix} a[1] & a[3] & & & & & & & & \\ & a[3] & a[5] & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & a[i-2] & a[i] & & & & & \\ & & & & a[i] & a[i+1] & & & & \\ & & & & & a[i+1] & a[i+3] & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & a[n-3] & a[n-1] & \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1+a[n-1] & \end{pmatrix}.$$

Then from Lemma 3.1 and Proposition 3.2 we obtain

$$\begin{aligned} \tilde{p}_i &= a[2, 4, \dots, i-1, i+2, i+4, \dots, n-2] \cdot \left[ a[3, 5, \dots, i, i+1, i+3, \dots, n-1] \right. \\ &\quad + \sum_{j=2}^{\frac{n-3}{2}} a[1, 3, \dots, 2j-1, 2j+3, \dots, i, i+1, i+3, \dots, n-1] \\ &\quad + a[1, 3, \dots, i-2, i+1, i+3, \dots, n-1] + a[1, 3, \dots, i, i+3, i+5, \dots, n-1] \\ &\quad + \sum_{j=2}^{\frac{n-i-2}{2}} a[1, 3, \dots, i, i+1, i+3, \dots, i+2j-3, i+2j+1, \dots, n-1] \\ &\quad \left. + a[1, 3, \dots, i, i+1, i+3, \dots, n-3](1 + a[n-1]) \right] \\ &= \frac{N(a)}{a[0, 1]} + \sum_{j=2}^{\frac{n-3}{2}} \frac{N(a)}{a[0, 2j+1]} + \frac{N(a)}{a[0, i]} + \frac{N(a)}{a[0, i+1]} \\ &\quad + \sum_{j=2}^{\frac{n-i-2}{2}} \frac{N(a)}{a[0, i+2j-1]} + \frac{N(a)}{a[0, n-1]}(1 + a[n-1]) \\ &= \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, i, i+1, i+3, \dots, n-1 \rangle \right). \end{aligned}$$

□

**Lemma 3.6.**

$$\tilde{p}_{n-2} = \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, n-2, n-1 \rangle \right).$$

*Proof.*

$$\begin{aligned}\tilde{p}_{n-2} &= \det \begin{pmatrix} 1+a[1] & 1 & \cdots & 1 & 1 & 1 \\ 1+a[1] & 1+a[2] & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a[n-3] & 1+a[n-2] & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1+a[n-1] \end{pmatrix} \\ &= \det \begin{pmatrix} & a[2] & & & & \\ a[1] & & a[3] & & & \\ & a[2] & & a[4] & & \\ & & \ddots & & \ddots & \\ & & & a[n-4] & & a[n-2] \\ & & & & a[n-3] & a[n-2] & a[n-1] \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1+a[n-1] \end{pmatrix}.\end{aligned}$$

By Laplace expansion we get

$$\tilde{p}_{n-2} = a[2, 4, \dots, n-3] \cdot \det \begin{pmatrix} a[1] & a[3] & & & & \\ & a[3] & a[5] & & & \\ & & \ddots & \ddots & & \\ & & & a[n-4] & a[n-2] & \\ & & & & a[n-2] & a[n-1] \\ 1 & 1 & \cdots & 1 & 1 & 1+a[n-1] \end{pmatrix}.$$

Then from Lemma 3.1 and Proposition 3.2 we obtain

$$\begin{aligned}\tilde{p}_{n-2} &= a[2, 4, \dots, n-3] \cdot \left[ a[3, 5, \dots, n-2, n-1] \right. \\ &\quad + \sum_{j=1}^{\frac{n-5}{2}} a[1, 3, \dots, 2j-1, 2j+3, \dots, n-2, n-1] \\ &\quad \left. + a[1, 3, \dots, n-4, n-1] + a[1, 3, \dots, n-2](1+a[n-1]) \right] \\ &= \frac{N(a)}{a[0, 1]} + \sum_{j=1}^{\frac{n-5}{2}} \frac{N(a)}{a[0, 2j+1]} + \frac{N(a)}{a[0, n-2]} + \frac{N(a)}{a[0, n-1]}(1+a[n-1])\end{aligned}$$

$$= \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, n-2, n-1 \rangle \right).$$

□

**Lemma 3.7.**

$$\tilde{p}_{n-1} = \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, n-2 \rangle \right).$$

*Proof.*

$$\begin{aligned} \tilde{p}_{n-1} &= \det \begin{pmatrix} 1+a[1] & 1 & \cdots & 1 & 1 \\ 1+a[1] & 1+a[2] & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a[n-2] & 1+a[n-1] \end{pmatrix} \\ &= \det \begin{pmatrix} & a[2] & & & \\ a[1] & & a[3] & & \\ & a[2] & & a[4] & \\ & & \ddots & & \ddots \\ & & & a[n-3] & & a[n-1] \\ 1 & 1 & \cdots & 1 & 1+a[n-2] & 1+a[n-1] \end{pmatrix}. \end{aligned}$$

By Laplace expansion we get

$$\tilde{p}_{n-1} = a[2, 4, \dots, n-1] \cdot \det \begin{pmatrix} a[1] & a[3] & & & \\ & a[3] & a[5] & & \\ & & \ddots & \ddots & \\ & & & a[n-4] & a[n-2] \\ 1 & 1 & \cdots & 1 & 1+a[n-2] \end{pmatrix}.$$

Then from Lemma 3.1 and Proposition 3.2 we obtain

$$\begin{aligned} \tilde{p}_{n-1} &= a[2, 4, \dots, n-1] \cdot \left[ a[3, 5, \dots, n-2] + \sum_{j=1}^{\frac{n-5}{2}} a[1, 3, \dots, 2j-1, 2j+3, \dots, n-2] \right. \\ &\quad \left. + a[1, 3, \dots, n-4](1+a[n-2]) \right] \end{aligned}$$



$$\begin{aligned}
&= \frac{N(a)}{a[0, 1]} + \sum_{j=1}^{\frac{n-5}{2}} \frac{N(a)}{a[0, 2j+1]} + \frac{N(a)}{a[0, n-2]}(1 + a[n-2]) \\
&= \frac{1}{a} \left( 1 + \frac{1}{a} \langle 1, 3, \dots, n-2 \rangle \right).
\end{aligned}$$

□

From Lemma 3.3–3.7, we can get the explicit representation of  $\tilde{P}^{-1}(x)$ . Finally, Theorem 2.2 follows from (1).

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